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SURFACE STRUCTURE OF AN INVARIANT MANIFOLD OF A HALO ORBIT

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Abstract

We extract the surface structure of the unstable invariant manifold tube projected into position space, of a halo orbit near L_2 in the circular restricted three body model. We do this by using transversal planes to intersect trajectories that approximate the tube. From these intersection points we construct spline-interpolated cross section curves which give a good idea of the structure of the tube. For example, we show that, for the value of μ we use, the tube pinches, develops a self-intersection, develops loop-inside-tube structure, pinches some more, and so on. We also construct surfaces made of quadrilaterals and triangles from these cross-sections. The transversal planes are obtained by taking planes orthogonal to a curve that follows the general shape of the tube. One such curve we use, is the unstable invariant manifold of the equilibrium point L_2 itself. In another example, we take a circle that follows the tube, as the curve for finding planes transversal to the tube. We also show that tubes of different energies, that start out in certain ordering, do not obey the ordering after a while. Our method is complementary to the method of taking cross-sections of constant time (the isochronous method), as used by some other researchers. The isochronous method is good at revealing the temporal structure of trajectories on a tube. However, due to the unequal speeds of different trajectories, it is harder to use for long length surface extraction. In contrast, using our method, we show cross-sections of the tube through an angular extent of nearly π during which the tube becomes extremely convoluted. Our work is motivated by applications to space mission design.

INTRODUCTION

In the spatial circular restricted three body problem (spatial CR3BP), halo orbits near L_2 have stable and unstable invariant two dimensional manifolds, forming two dimensional tubes in six dimensional phase (position-velocity) space. See the thesis of Ross [2004] for

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background information about CR3BP, invariant manifolds in phase space of CR3BP, and the use of these invariant manifolds for space mission design. These tubes can be projected into three dimensional configuration (position) space to form two dimensional surfaces embedded in $\mathbb{R}^3$. What is the structure of such surfaces ? There appears to be no complete answer to this question in the literature. A common way to understand the structure of these surfaces is to linearize the equations of motion about a periodic orbit, then use eigenvectors of the linearization to create trajectories along the stable or unstable directions. See for instance Ross [2004]; Gómez et al. [2004]. However, when the trajectories are viewed, even with an interactive 3D viewer, the ambiguities in the shape of the tube that they represent, are not easy to resolve. Figure 1 shows one branch of the unstable manifold of a halo orbit. We now know from the use of our method that the cross-sections of this tube pinch, the tube develops a self-intersection and then develops further complicated structure downstream from the halo orbit. This is not apparent in Figure 1, which is the usual way of visualizing such tubes that is popular in the literature. Note that 100 trajectories have been drawn to approximate the tube but the structure mentioned above is still not apparent in Figure 1. With our technique this structure is immediately visible. See Figure 2, and 3 for an example of the use of our method. Figures 5,6, 7 explain our method in more detail and Figures 4,13 show the tube surface made of quadrilaterals and triangles, and created from the cross-sections obtained by our method.

Notation: In this paper we use the convention of placing the smaller primary at $(1-\mu, 0)$ and the larger one at $(-\mu, 0)$.

Motivating Applications

In design of low thrust trajectories for space missions, stability of the trajectory during loss of thrust is an issue. If stable manifolds are used to approach the bodies, as long as they do not impact a body, they will lead to capture during loss of thrust. In order to obtain initial conditions that put the spacecraft on such trajectories, one needs to know the phase space structure. In addition, the knowledge of phase space structure can be useful for designing approximate capture or escape trajectories quickly, which can then be optimized and refined using optimizers which use the full ephemerides and detailed force models to model spacecraft motion.

Previous Work

The earliest work we are aware of is the unpublished report Thurman and Worfolk [1996] in which the surface is extracted for short lengths of the tubes. The only published work we know of in this subject is Howell et al. [2004]. However, in the comparable part of their work they use constant time sections to obtain a surface. We will call this the isochronous method in what follows. Our method is complementary to the isochronous method of Howell et al. [2004] in the sense that we use spatial sectioning while they use isochronous sectioning. Both methods have their strengths and weaknesses. For example, the isochronous

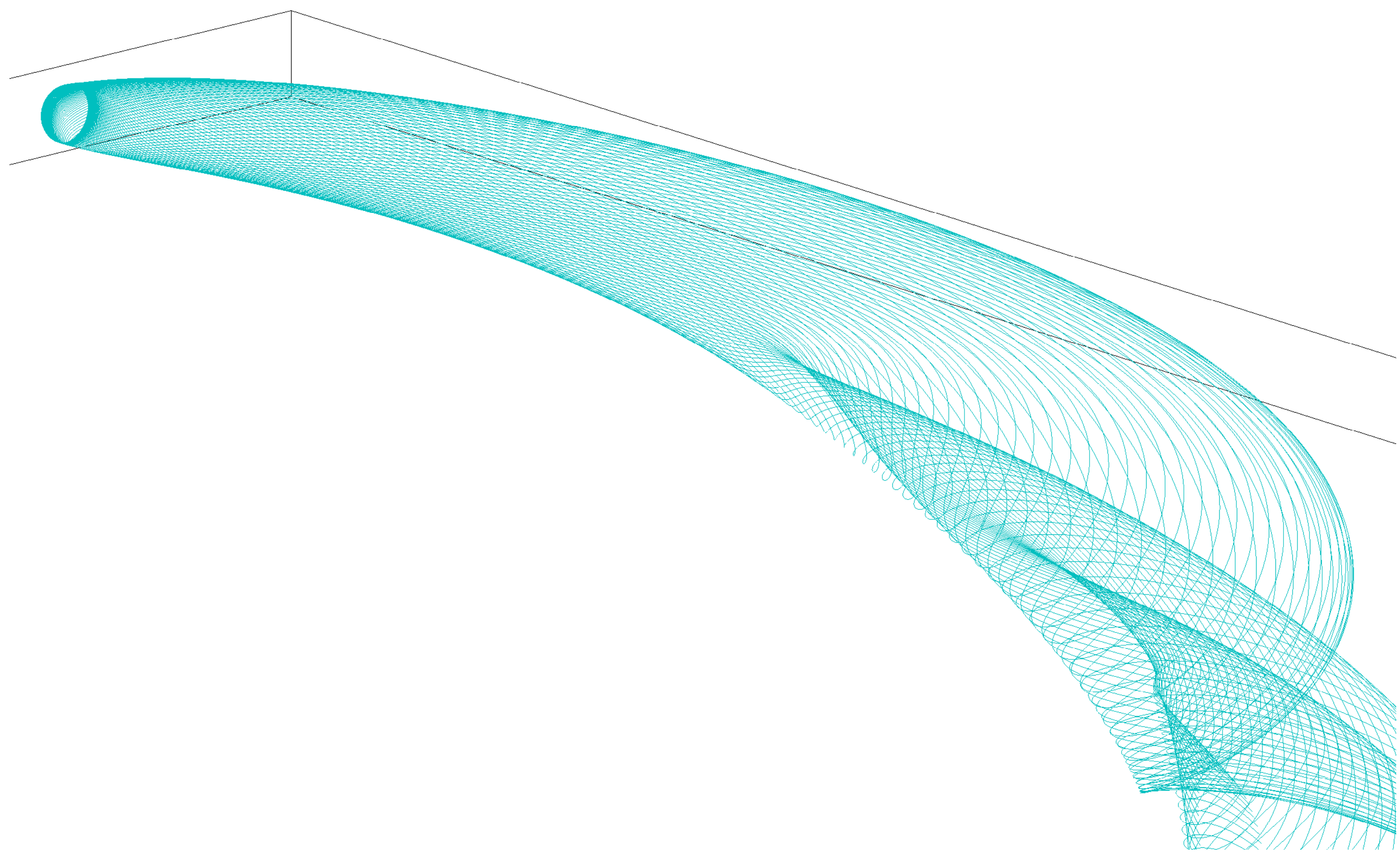

Figure 1: Illustration of a commonly used method found in literature, for visualizing tubes. Here 100 trajectories approximating the unstable outward going invariant manifold of a halo orbit near L_2. The dark parallel lines in the background give an idea of the vertical dimension and viewing angle. Notice that the pinching and development of a self-intersection is not apparent in this way of visualizing the tubes. The other structure of loop-inside-tube is also not unambiguously visible. Rotating and viewing the trajectories interactively using a 3D viewer is only marginally more effective. Our technique however, immediately makes the structure obvious as can be seen in Figure 2 and even better, in Figure 3. For this figure, Jacobi constant $C \approx 3.00052$ and $\mu \approx 3.04 \times 10^{-6}$. This value of μ corresponds to the mass of Sun as the primary mass and mass of Earth plus mass of Moon as the secondary mass.

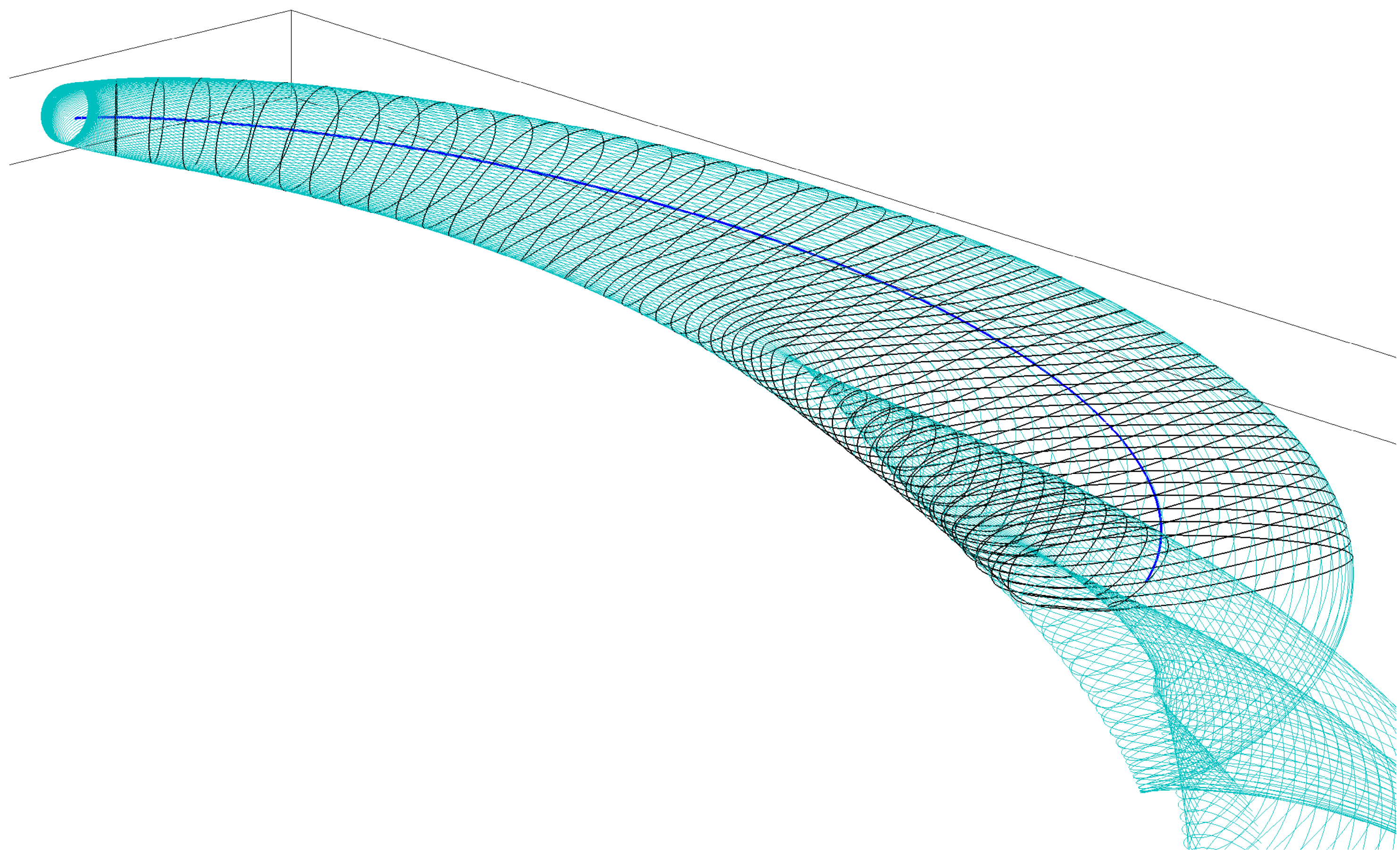

Figure 2: Illustration of our method of visualizing tubes. Using our method of computing cross-sections, one can see the pinching, the figure-8 self-intersection, and the loop-inside-tube structure. Figure 3 shows the cross-section curves individually making the structure even more apparent. A tube surface assembled from the cross-sections and made from quadrilateral patches is shown in Figure 4. The cross section black curves in the current figure are spline interpolations of intersections points obtained by intersecting transversal planes with the trajectories approximating the tube. Without these the structure of the tube is not easily visible, as can been verified by viewing Figure 1. The thick blue line in the middle is part of the unstable manifold of the equilibrium point L_2. It was used as an automated way of obtaining the transversal planes since it follows the general direction of the tube. In this case the planes, used for obtaining the cross-sections, were placed orthogonally to this curve. Other curves could also have been used just as easily, for example, part of a circle. Analogous result obtained by using a circular arc are shown in Figure 9 and 10. See also discussion in Section . For this figure, Jacobi constant $C \approx 3.00052$ and $\mu \approx 3.04 \times 10^{-6}$. This value of μ corresponds to the mass of Sun as the primary mass and mass of Earth plus mass of Moon as the secondary mass. For more details on our method see Figures 5,6,7.

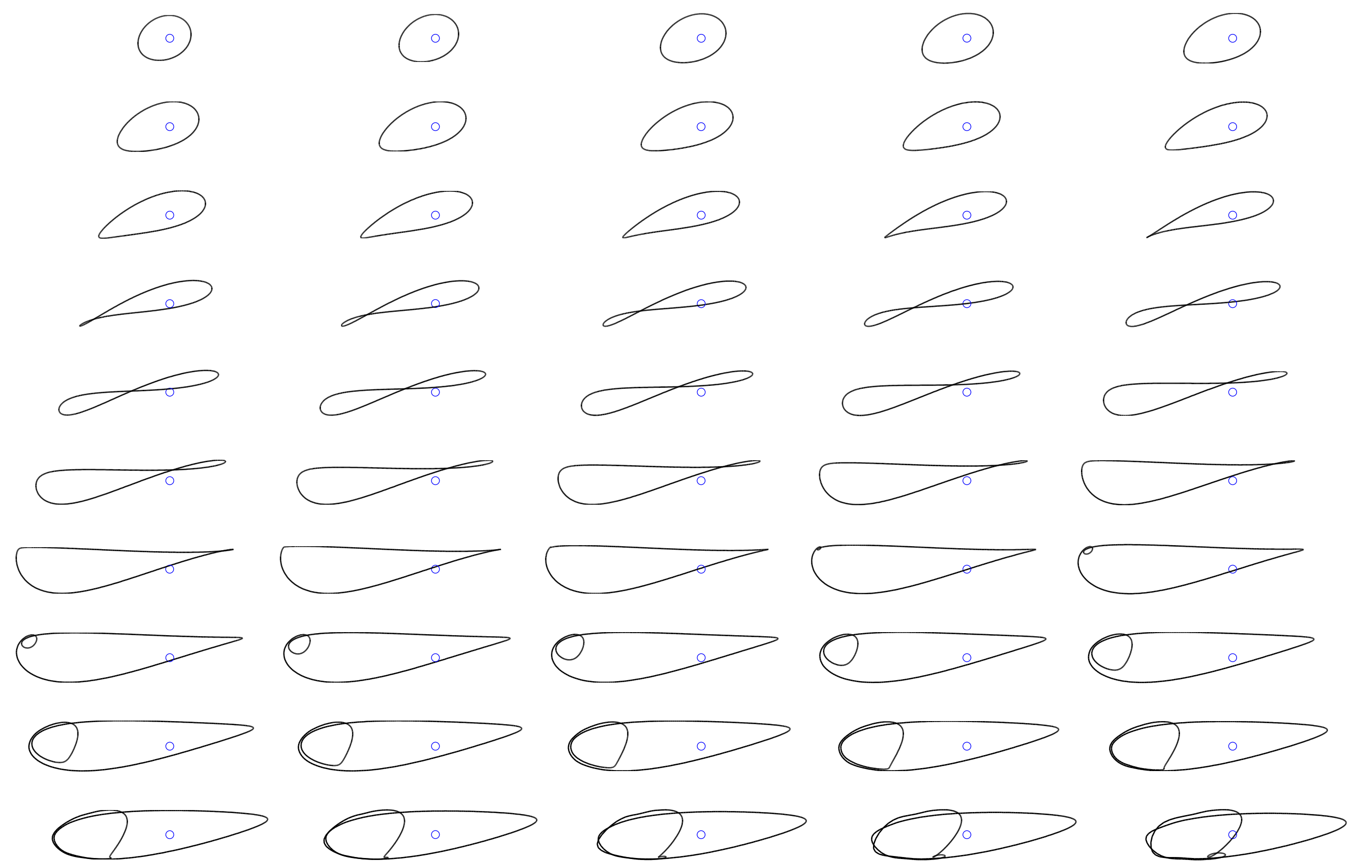

Figure 3: Another way of visualizing the tube structure. The cross-section curves of Figure 2 are shown individually in front views. They are arranged left to right, top to bottom, as one moves downstream along the tube from the halo orbit that generated the tube. These curves are all viewed from an identical distance along the normal from the origin of the plane in which that curve lies. They are also drawn to scale, so the size difference amongst the curves is actual. The small blue circles are the origin of the plane in which that curve lies. It also represents the location of the unstable manifold of L_2, which was used as the curve along which the planes were placed orthogonally. See Figures 5,6 for illustration of this point. The current figure also shows that the unstable manifold of L_2 does not stay inside the tube (not that there was any reason to expect that it would), another example of the types of structures and their relationships that can be discovered using our method. In Figure 9 we show the cross-sections much further downstream in the tube.

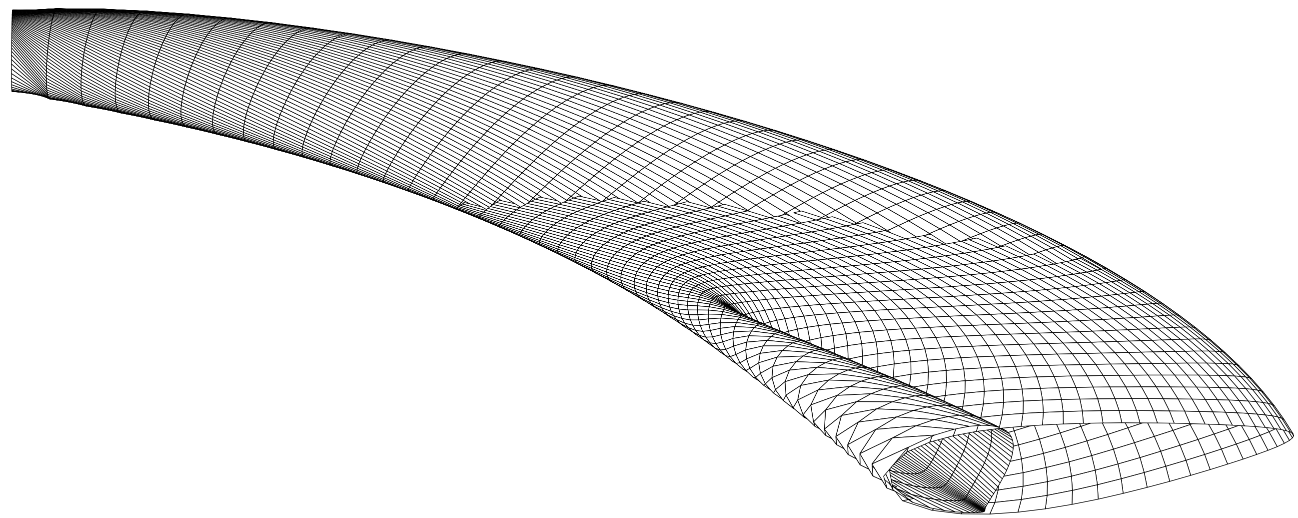

Figure 4: Example of a surface assembled from cross-sections obtained using our method. This surface is made from 100 trajectories. The cross-sections obtained using our method, and shown in Figure 2, are interpolated as piecewise-affine curves (instead of being interpolated by cubic splines as in Figure 2). Points corresponding to the same trajectory on adjacent piecewise-affine cross-section curves are joined by lines to obtained quadrilateral patches. Since the patches need not be planar, each quadrilateral can be split into two triangles as shown in Figure 13. An adaptive method can improve the surface substantially as discussed in Section but has not been implemented. See also the comments about distortion from twisting of trajectories, in the caption of Figure 13. Further downstream it makes less and less sense to compute such surfaces since the tube gets very convoluted as shown in Figure 9. The cross-section curves however, continue to be a useful tool even in such portions of the tube. (The error in hidden line and hidden surface removal in these surfaces near the self-intersections is a limitation of the rendering program used).

method exposes the temporal structure of the trajectories on a tube. On the other hand long length surface extraction is harder with that method, precisely because of this strength in the temporal domain. See Figure 11 to see the distortion that the isochronous method introduces. See also more discussion in Section .

Our Results

We use transversal planes to intersect the trajectories approximating the tube. The intersection points are then interpolated using cubic splines or lines. This gives a very good idea of the shape of the tube. For example, with this tool we have discovered that the tube becomes pinched, then self-intersects so that the cross-sections develop a figure-8 shape. Then as one moves downstream, the tube becomes pinched on the opposite side and then becomes a double loop which develops more pinching and so on. The tube becomes more and more convoluted (rolled up) as one moves downstream. All this can be seen in Figures 2,3 and 9. Surfaces assembled from the cross-sections obtained by our method are shown in Figures 4 and 13. Furthermore, we have discovered that the invariant unstable manifold of L_2 (which we will call L_2W^u) does not stay inside the tubes of certain energies (there was no reason to assume that it would anyway), while it follows the general shape of the unstable manifold going away from the secondary body. We have also discovered that tubes of 3 different energies, that start out one inside the next, do not maintain this ordering in certain places. This is clear in Figures 14 and 15. These are examples of the types of structures and relationships between structures that one can discover using our method. In the results of our experiments, it is not clear, how much the properties of the integrator used for propagating trajectories affects the results. The use of interval arithmetic, analytical techniques, and perhaps variational integrators might help to clarify whether the details of the results are numerical artifacts. This is something we have not done.

SURFACE STRUCTURE OF TUBE

Isochronous cross-sectioning

One method for obtaining the surface structure that has been considered by Howell et al. [2004] is to take isochronous cross-sections. This means, one considers points on all the trajectories that take the same amount of time to be reached from the halo orbit, along their respective trajectory. This set of points gives an isochronous cross-section (not necessarily planar) which can then be used to build a surface of the tube. One strength of this isochronous method is that it is able to reveal the temporal structure of the solutions. However this also becomes a handicap in long length surface extraction because the range of speeds of the trajectories is so large that some trajectories are left far behind others. Thus the isochronous cross-sections can become very stretched along the tube. This is more readily understood by viewing Figure 11. For short lengths of the tube, one can use these cross sections and join points of same trajectories on adjacent cross-sections to make quadrilateral patches. The surface made with quadrilateral patches, and extracted using this technique is shown in Figure 12. Since the trajectories on the tube wind around a lot, In addition to the stretching along the length of the tube, caused by the non-uniform motion of trajectories, there is another distortion visible in Figures 11 and 12. This is that the patches forming the surface become distorted in a direction around the short dimension of the tube. Thus, the use of isochronous cross-sections to extract surfaces suffers from distortions of the patches, both along and around the tube. Our technique described in this paper addresses the distortion along the length of the tube, and lays the groundwork for removing the distortion around the short dimension of the tube.

Sectioning with transversal planes

Instead of using isochronous cross-sections we use planes transversal to the tube to obtain the cross-sections. Our method is summarized in Figures 5,6, 7 and 8 and some results are in Figures 2, 3 and 9. While the planes could have been placed manually, in our work we use some guiding curve, called the *core* curve in this paper, along the tube to place the transversal planes. We place the planes such that the inter-plane distance along the curve is roughly equal. See, Figure 6. The intersection of planes with the trajectories approximating the tube are then computed and the intersection points interpolated, as shown in Figure 5. Figure 7 shows the top view of a single trajectory intersecting planes that have been placed along, and transversal to a core curve. We only retain the first intersection with a plane. The part of the trajectory that precedes this intersection is discarded when considering the next plane. See Figure 7 for illustration of this point. This choice is made by our algorithm because when we interpolate the intersection points (of trajectories with a given plane) into a cross-section curve, we do so in the order in which the trajectories were when they started. Thus each trajectory can appear as only one intersection point in a plane, in order for the sequence to make sense.

Intersection of planes amongst themselves can sometimes be a problem. The problem may exist if the trajectories on tube intersect beyond the intersection of the planes. This can result in gaps in tube surface during reconstruction, because of the way our algorithm works. In the tube going outwards from L_2, use of a circle as a core curve can solve the problem of transversal planes intersecting amongst themselves, if it exists. The intersection then will be at the center of the circle, which is far away from the trajectories. We have shown this set up in Figure 10. More sophisticated plane placement can be done, which allows planes to become non-orthogonal to core curve, allows them to adjust along core curve, or allows core curve to straighten out, in order to push the intersections away from the danger zones. We have not implemented these more sophisticated algorithms for plane placement since the use of a circle in this case, is an easier solution.

Surface structure

The intersection points of trajectories approximating the tube with a given plane can be interpolated in any suitable manner. We have shown cubic spline interpolation in Figure 5 and 2. Front views of these planar cross-section curves are shown in Figures 3, 8 and

9. These cross-sections give a very good idea of the surface structure of the tube. We also use a piecewise affine interpolation of the points of intersections of trajectories with planes to obtain surfaces made of quadrilaterals or triangles. The quadrilateral are obtained by taking two adjacent piecewise-affine cross-section curves and joining pairs of points corresponding to the same trajectory. The triangles are then obtained by triangulating each quadrilateral. Surfaces obtained by this method are shown in Figure 4 and 13.

This method of surface construction can be applied to sections of the tube longer than that achievable with isochronous cross-sections of Figures 11 and 12. However applying it to something like an angular extent of π of the tube shown in Figure 10 would be of questionable value. One can see in the front views of the cross-section curves in Figure 9, that the tube becomes very rolled up inside itself as one travels downstream along the length of the tube. A surface made from patches in such portions of the tube would not be very easy to interpret visually. The cross-section curves however, continue to be a useful tool to study the surface structure even when the tube becomes so convoluted.

There are several approaches possible for improving the quality of the cross-sections and surfaces with fewer trajectories, by doing adaptive sampling. This is discussed more in Section .

SUMMARY OF RESULTS

Using the algorithms described in this paper we have studied the surface structure of the projection into configuration (position) space, of an invariant manifold. The manifold is a surface in $\mathbb{R}^3$ and it is one branch (going away from the secondary) of the unstable manifold of a halo orbit around the equilibrium point L_2 in the spatial CR3BP. We used $\mu \approx 3.04 \times 10^{-6}$ which corresponds to the mass of Sun as the primary and the mass of Earth and Moon as the secondary body. Most of our figures are for a Jacobi constant $C \approx 3.00052$. Our results are as follows:

(i) The tube-like surface pinches, self-intersects and then develops complicated rolled up structure further downstream. This is shown in Figure 2, 3 and 9.

(ii) Tubes at 3 different energy levels that start out one inside the other near the halo orbit, do not maintain that ordering further downstream. This is shown in Figure 14.

(iii) The invariant manifold of L_2 while following the general shape of the tube, does not stay inside it. This is apparent in Figure 3.

(iv) Cross-sections we find by our method can be used to construct tube surfaces made of quadrilaterals or triangles as shown in Figures 4 and 13. These do not exhibit the distortions along the length of the tube, that are apparent in surfaces made from isochronous cross-sections shown in Figures 11 and 12.

(v) The cross-section method allows us to understand the surface structure a long distance away from the halo orbit, as shown in Figures 10 and 9 even after the surface has become very convoluted and rolled up.

(vi) The cross-sections obtained by our method provide the capability to remove the distortion in surface patches caused by the trajectories winding around the tube. This is discussed more in Section in the paragraph on adaptive sampling.

SHORTCOMINGS AND FUTURE DIRECTIONS

Dependency on parameters and numerical methods: We have not studied the dependency of core curve or the trajectories, on the exact position from which they are started out (the initial positions are offset a little bit from where the linearization is performed). Dependency on the tolerances and properties of the numerical integrator have also not been investigated. Variational integrators, such as symplectic integrators, are known to preserve the qualitative properties of the phase portrait even with large time steps. Thus it might be worthwhile to try variational integrators to see if they provide for greater accuracy with less computational effort. For an introduction to variational integrators see Marsden and West [2001].

Structure very close to the halo orbit: Close to the halo orbit, the trajectories move much faster around the tube than along the tube. Thus planes transversal to the core curve are nearly parallel to the trajectories. This makes for a very poor sampling in the region very close to the orbit. In this paper we have started a little distance from the halo orbit. Some other technique would have to be used to obtain the surface structure very close to the halo orbit. Perhaps the use of planes radial to the tube might be more appropriate there. Another possibility for this region is to use some method that constructs a surface from a cloud of points mentioned later in this section.

Structure in more complicated parts of phase space: The unstable manifold of L_2 is very well behaved in the portion of phase space we have studied. If however one travels in the opposite direction, towards the secondary body, the L_2W^u quickly becomes tangled around the secondary body before escaping into a nice trajectory. Thus it would not be of much use a as a core curve in the tangled region, except perhaps for short portions of the tube.

Resampling of cross-section curves: Since the trajectories wind around the tube, the patches making up the tube surface get somewhat distorted. To avoid this we suggest the following technique which we have not implemented yet. The cubic spline cross-section curves can be resampled at equal arc length intervals, starting from the top most point and it is these points that can be joined in adjacent cross-sections to obtain quadrilateral patches. All the cross-section curves are in planes that are perpendicular to the xy-plane. This resampling suggested above should get rid of the twisting caused by the around-the-tube twisting of the trajectories which is apparent in Figure 1.

Reconstruction from cloud of points: A completely different method for surface construction might be to consider all the points on the trajectories as a cloud of points and apply algorithms from computer graphics literature for constructing meshes from clouds of points. In fact we know more than just the cloud of points, since we know the velocity, energy and we know which points lie on the same trajectory. This extra information may be useful in inventing an algorithm for surface reconstruction. In any case, such algorithms will probably not work in the regions far downstream where the surface gets very convoluted as shown in Figure 9. But it should be possible to use the method in the early parts of the tube. It is also possible to investigate other surface reconstructions, based on plane-trajectory intersections, where all the intersections are collected as data. Then one may apply algorithms for obtaining curves from clouds of points lying in a plane. Clustering algorithms for example might be good candidates here.

Adaptive sampling: In some cross-sections, the intersections of the trajectories with that plane are not uniformly separated, some points being much further apart than others. Consider two such points that are far apart but come from trajectories that are logically adjacent to each other. To fill in between them, one needs to start out trajectories between them. One can go back to the halo orbit and since all the velocities and positions are known on it, one can compute the initial conditions for trajectories to start between the two in question. The intersections of these don't even have to be recorded for the earlier planes where the point distribution of intersection points is uniform. This provides an adaptive strategy for obtaining high quality cross-section curves with fewer trajectories.

ACKNOWLEDGMENTS

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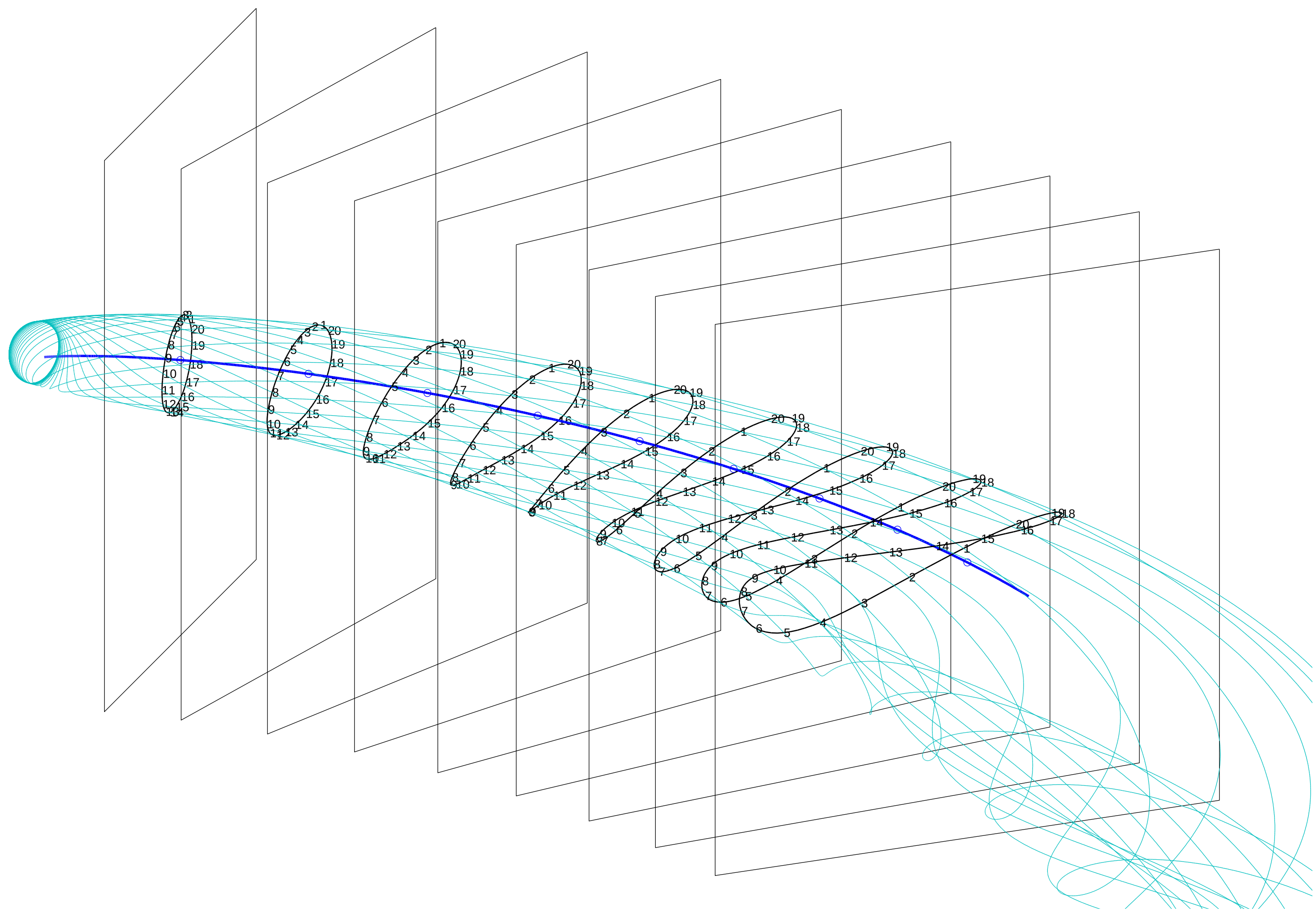

Figure 5: An explanation of our technique for obtaining cross-sections of the tubes to obtain surface structure. The thick blue curve (called the *core* curve) is used for placing planes transversal to the tube, spaced equally by arc length along the core curve. The planes, which are orthogonal to the core curve in this figure, are shown as transparent squares here. The points where the planes intersect the trajectories are then interpolated. Here the interpolation is by cubic splines and the intersection points have been labeled by the trajectory number that they belong to. Figure 7 illustrates the algorithm for intersecting a single trajectory with transversal planes. In the current figure, the unstable manifold of L_2 is used as the core curve although any curve that follows the general shape of the tube can be used. Some of the cross-section curves are shown in front view in Figure 8. Figure 6 shows a top view showing the plane placement. Figures 2 and 3 show the cross-section curves obtained for this tube when 100 trajectories are used to approximate the tube. Figures 4 and 13 show the surfaces assembled from the cross-sections using quadrilateral and triangular patches.

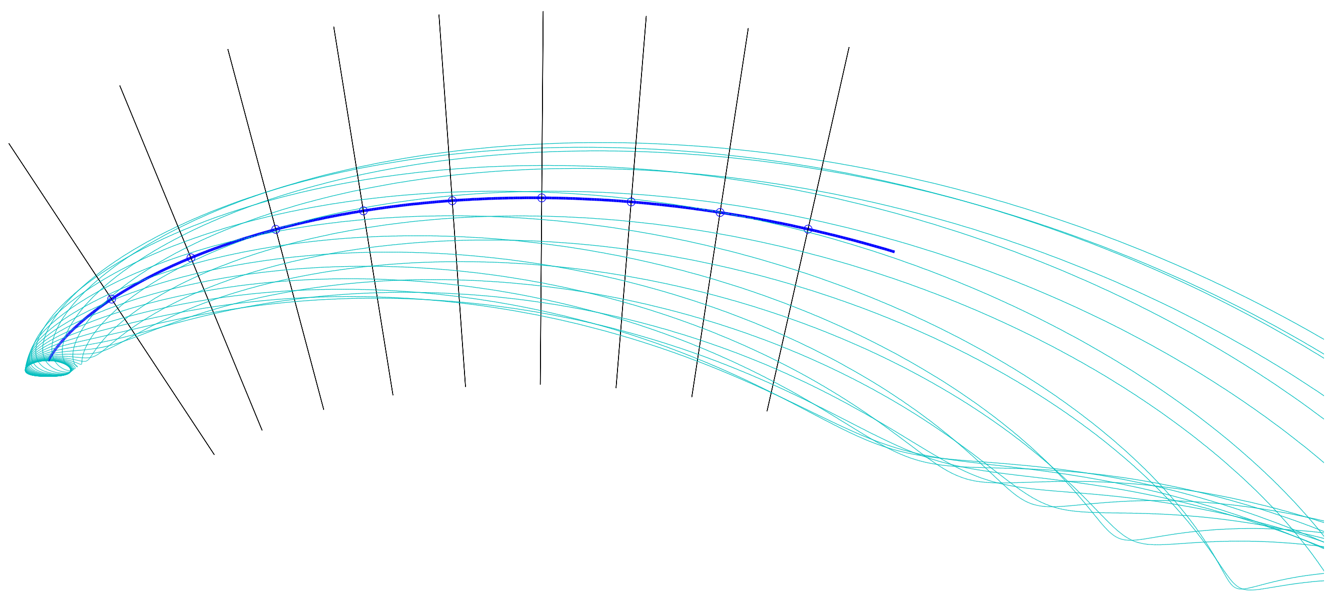

Figure 6: Top view of the configuration in Figure 6. The planes which are used for obtaining the cross-section curve appear as lines orthogonal to the core (thick blue) curve in this view. Some of the cross-section curves are shown in front views in Figure 8. The core curve used here is the unstable manifold of L_2 although any curve that follows the general shape of the tube could have been used. Indeed results from using a circle as the core curve are shown in Figure 9.

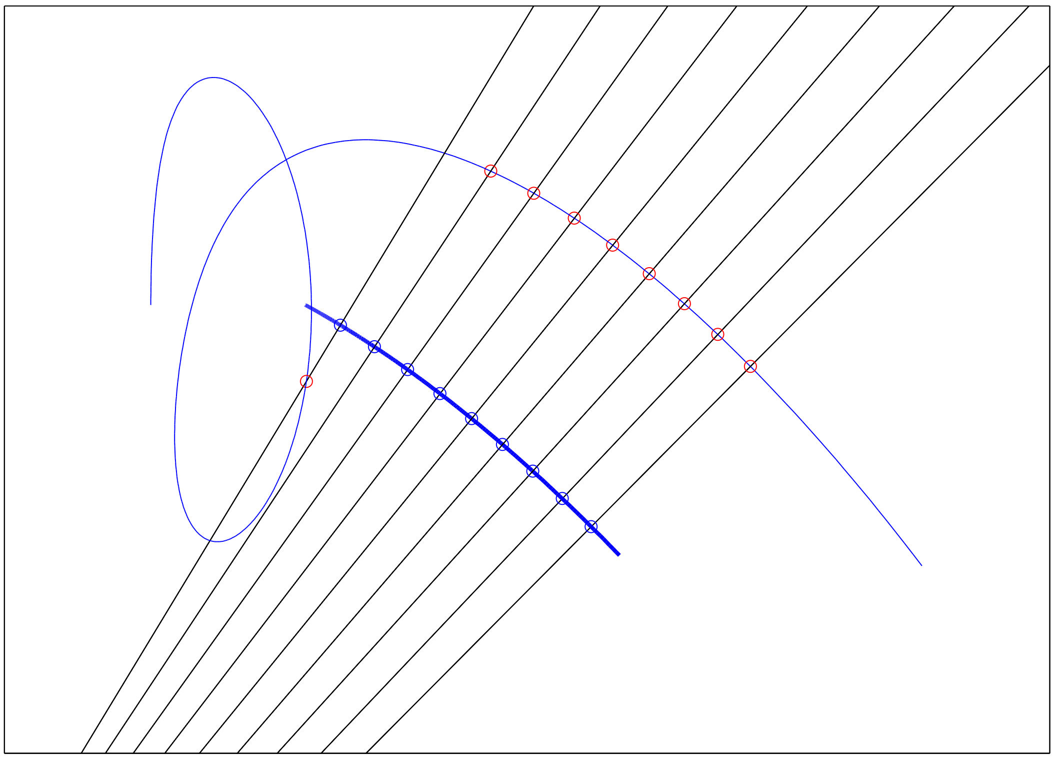

Figure 7: Top view of intersection of planes with a single trajectory. Planes appear as straight lines in this top view. The trajectory is the thinner curve and the thicker curve is the core curve used for obtaining transversal planes. See caption of Figure 5 for more details. A red circle marks the intersection of trajectory with a plane. Blue circles are locations of plane origins along core, thick blue curve. Trajectory intersects the first plane 3 times, but as shown, we only use the first intersection and ignore the subsequent intersections with same plane. This is required by our algorithm for obtaining cross-sections. We join the intersections of all trajectories with a plane, in sequence, to get the cross-section. Thus each trajectory has to appear as only one intersection point in a plane, in order for the sequence to make sense.

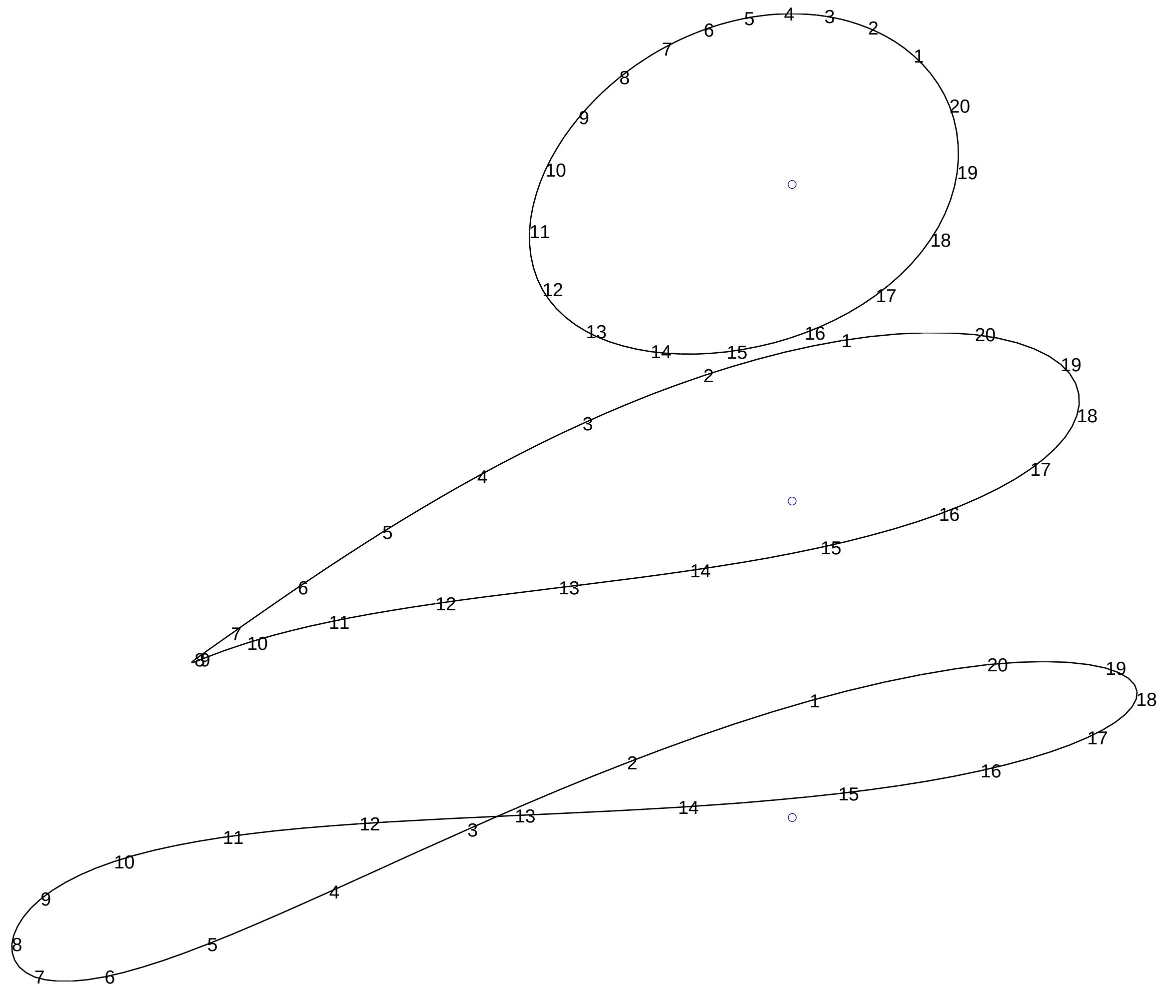

Figure 8: Close-up, front view of some of the cross-section curves of Figure 5. The curves are the curve numbers 1, 5 and 7 of Figure 5, counting from near the halo orbit (the small end of the tube). The numbers mark the intersection points of the trajectory labeled by that number and the plane in which the cross-section curve lies. Figure 3 shows many such cross-section curves in front views, using 100 trajectories to approximate the tube (instead of the 20 used here).

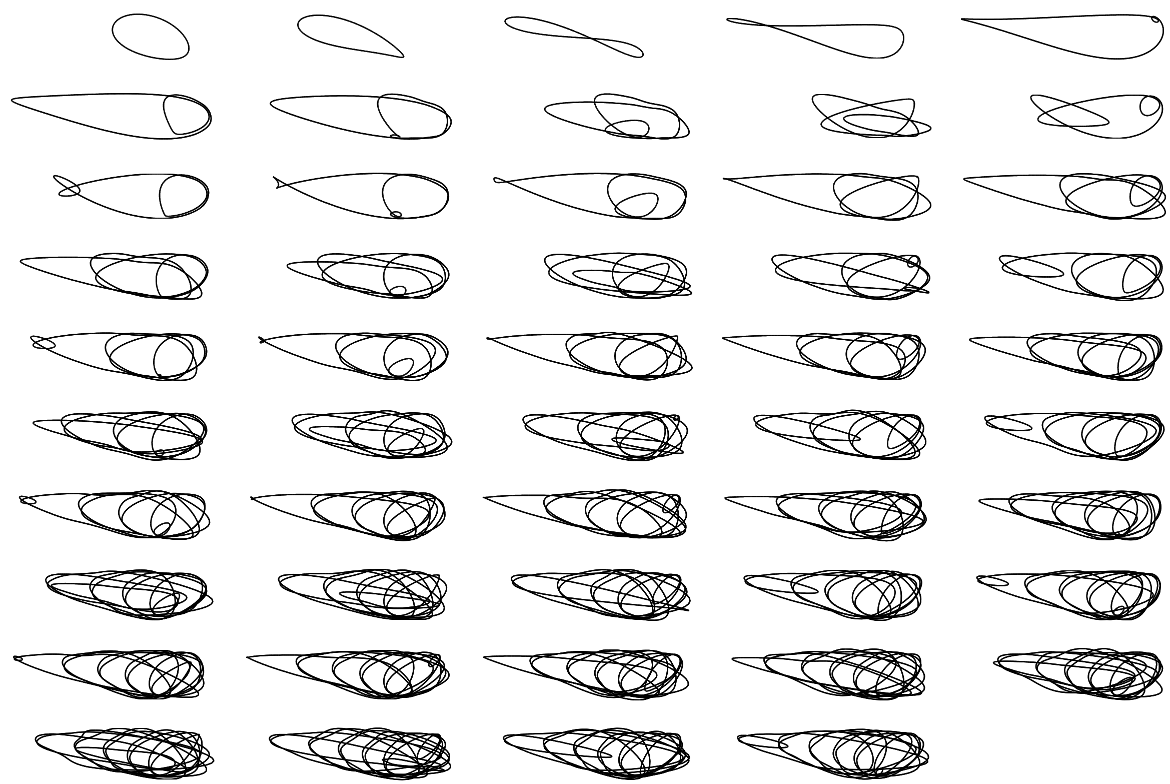

Figure 9: Analogous result to that shown in Figure 3. These cross-sections were generated from 100 trajectories, by creating planes orthogonal to a circle and using those as the planes transversal to the tube. See Figure 10 to see the top view of the configuration used to generate these cross-sections. The current figure also shows that computing an actual surface from the cross-sections has questionable value far away from the start of the tube, since the tube becomes so convoluted at some distance downstream from the halo orbit. The cross-sections themselves however, remain a powerful tool for understanding the convoluted structure which would have been completely invisible in the standard technique used in literature, of visualizing tubes by drawing trajectories as shown in Figure 1. We also think that taking isochronous cross-sections (Figure 11) so far down the tube would not have been illuminating due to the resulting extreme distortion of the cross-sections. The advantage of using arbitrary curves as the core curves (rather than being limited to the unstable manifold of L_2) is that in this case of using a circle for example, the planes intersect only at the center of the circle. Recall that intersection of planes can sometimes cause problems in computing cross-sections as discussed in Section .

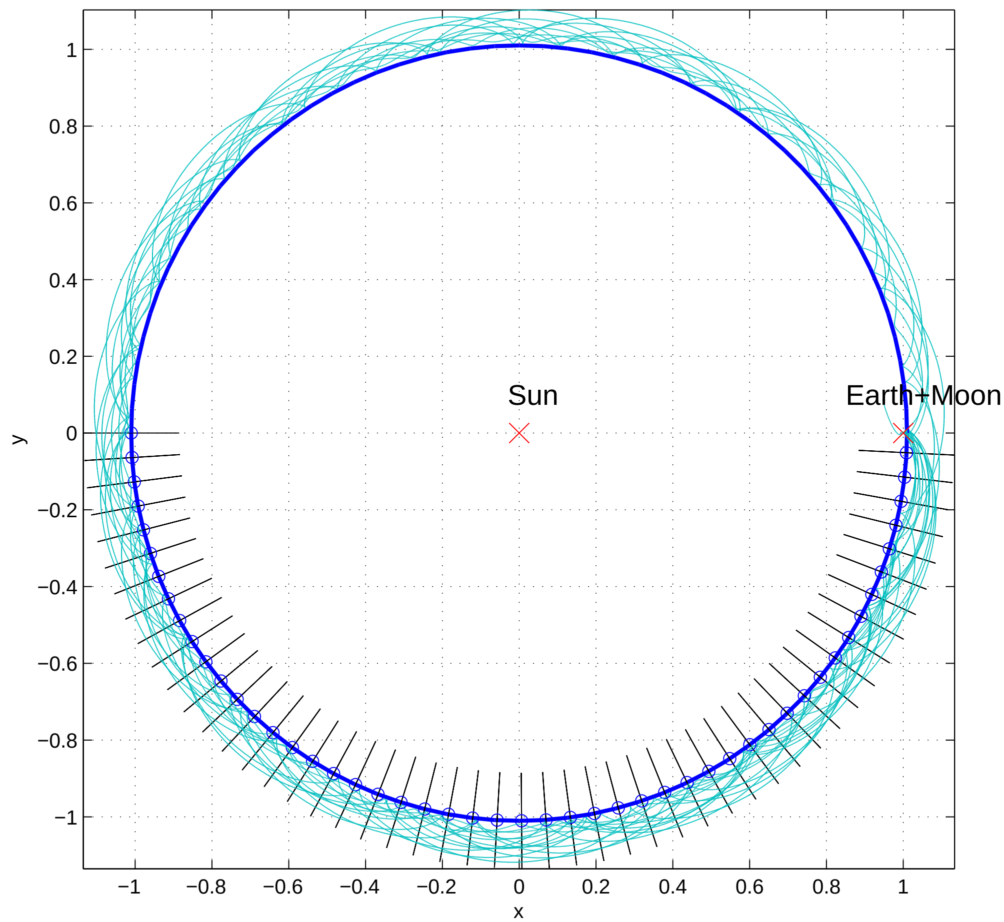

Figure 10: Using other curves to obtain transversal planes. This setup was used to generate the cross-sections shown in Figure 9. The thick blue curve is a circle centered at $(0, 0, 0)$ and it is used to generate the planes which appear as radial lines in this top view. Obviously, a circle of any radius would do the job. For illustration we show the circle of radius equal to the distance of L_2 from the origin. The primary body in this is Sun and the secondary body is a body with mass equal to the total mass of Earth and Moon. The sun is at $-\mu$ and the earth-moon is at $1 - \mu$. We have shown only 10 of the 100 trajectories (shown in cyan) for illustration. Using a circle in this case has the advantage that the planes do not intersect except at the center of the circle, which is far beyond where the trajectories lie. Thus we are able to catch all the intersection points which would be missing (due to the algorithm used) if the planes intersected amongst themselves and the trajectories lay beyond the intersection points. This point is discussed more in Section .

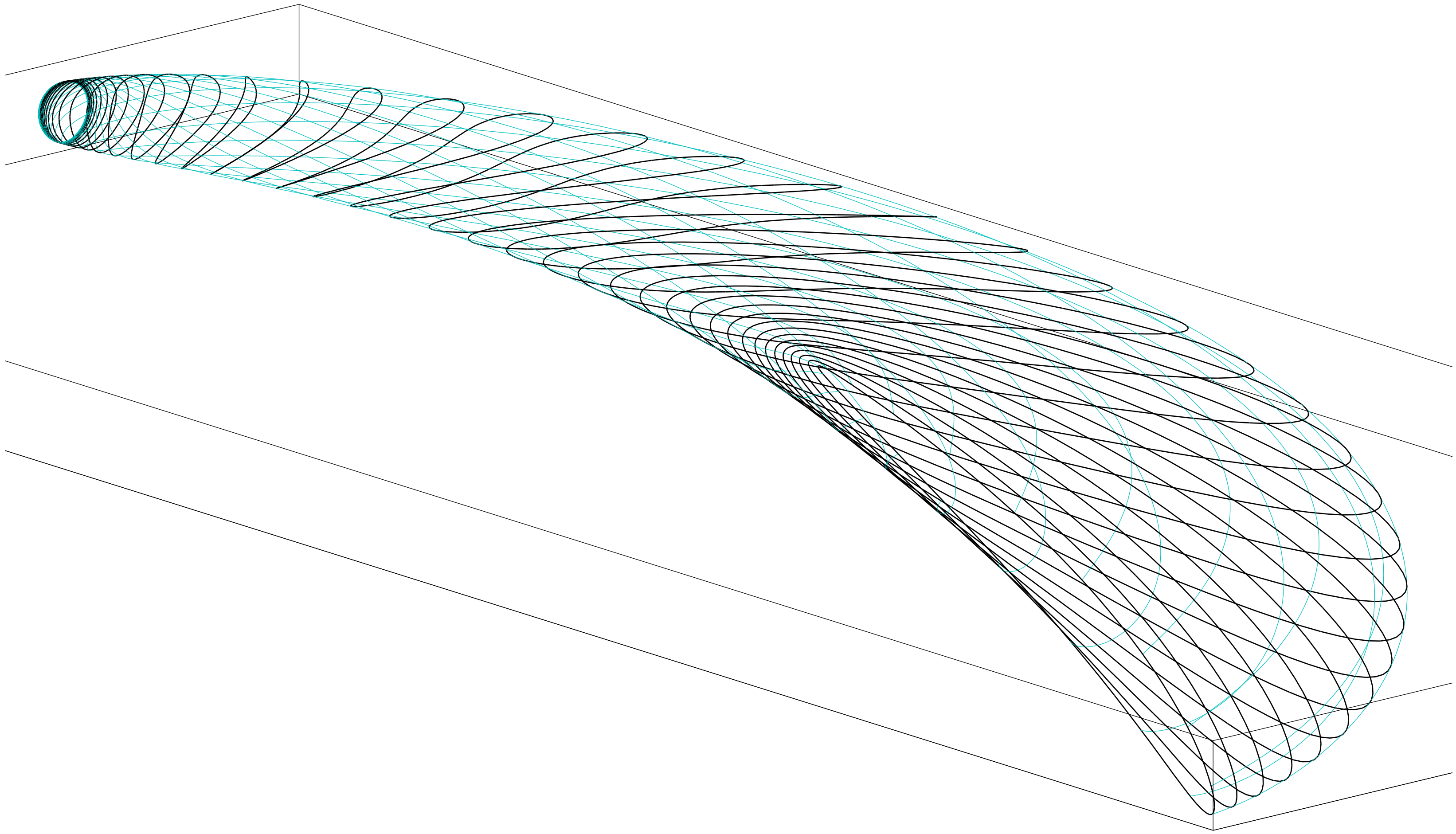

Figure 11: An alternative technique (the isochronous method) for obtaining the surface, based on ideas in Howell et al. [2004], shown here for comparison with our method. This method is good at revealing the temporal structure of the trajectories on tube. The black lines, transversal to the tube are isochronous cross-sections. They are obtained by taking points on the trajectories that take the same amount of time starting from near the halo orbit, and joining those points to make a curve. In this figure we have interpolated the points with a cubic spline. Although such cross-section curves can be used to construct a surface of the tube, the distortion makes it challenging to do this for longer portions of the tube. This is because points of different trajectories that start out at the same time near the halo orbit, travel at dramatically different speeds. This should be readily apparent in the stretching and non-uniform downstream turning of the isochronous cross-sections. Compare this to the cross-sections obtained by our method, shown in Figure 2. Also, since the cross sections in this figure are not planar curves, deciding what is a real self-intersection versus what is an illusion due to the viewpoint, is not easy. Only 20 trajectories are used in the current figure, because it is the non-uniform downstream stretching behavior of the cross-sections that is important here. Linear approximations of the cross-sections are assembled into a surface made of quadrilateral patches in Figure 12.

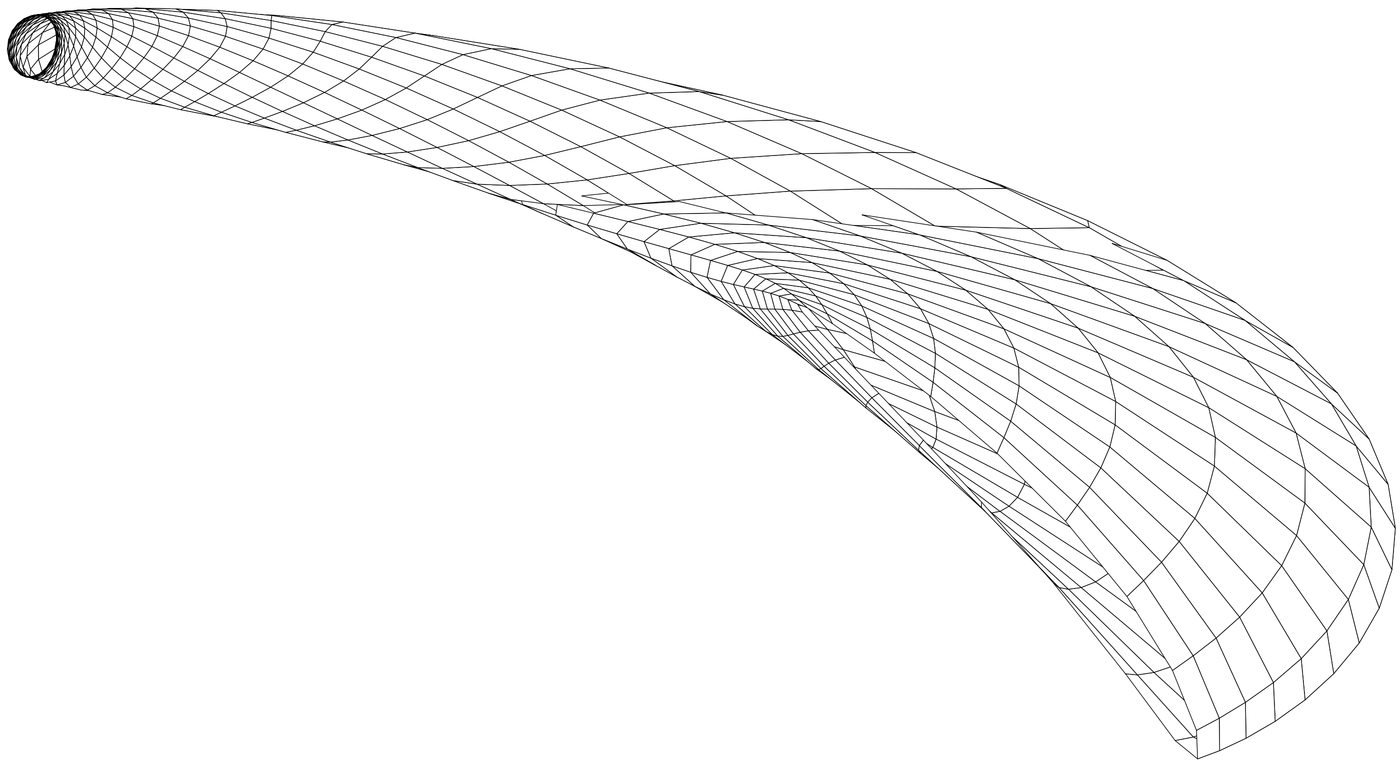

Figure 12: Surface made from cross-sections using the alternative technique based on ideas in Howell et al. [2004]. Here 20 trajectories are used and isochronous cross-sections computed as in Figure 11, but interpolated as piecewise affine curves rather than piecewise-cubics shown in Figure 11. Points corresponding to same trajectories are connected in adjacent cross-sections to obtain quadrilateral patches. Such quads need not be planar so it would have been better to make 2 triangles from each patch. We leave it as quad patches to show clearly the distortion that comes from the winding of the trajectories around the tube. The other distortion comes from the fact that points on different trajectories travel at different speeds as shown in Figure 11. Our method described in this paper addresses the first distortion and lays the groundwork to address the second. The cross-sections obtained by our method are in Figures 2, 3. Surfaces made of quadrilateral and triangular patches and assembled from the cross-sections of our method are in Figures 4,13.(The errors in hidden line and hidden surface removal in these surfaces near the self-intersections is a limitation of the rendering program used).

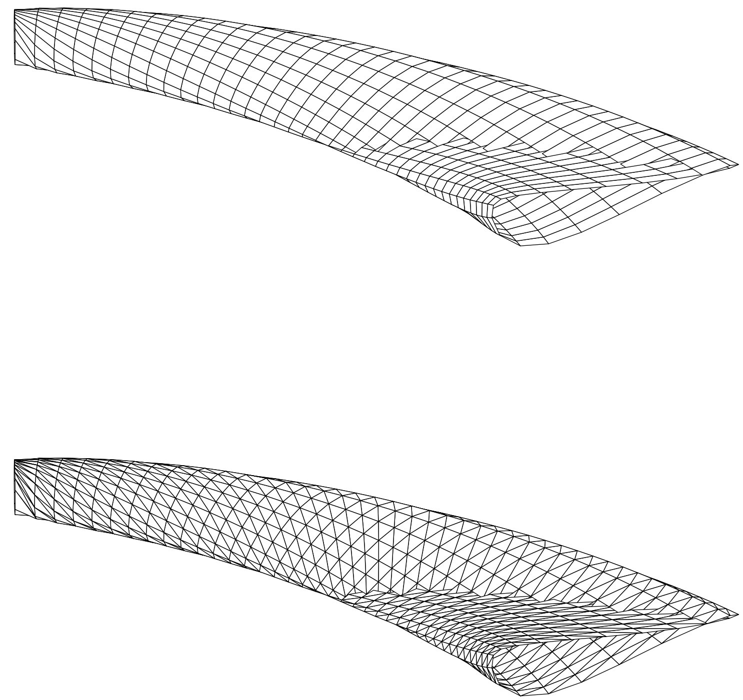

Figure 13: Preliminary surfaces made of quadrilateral and triangular patches from cross-sections of our method. This surface is made from 20 trajectories by taking planar cross-sections using our method and interpolating the cross-sections as piecewise-affine curves. In the top figure points corresponding to same trajectories are connected in adjacent cross-sections to obtain quadrilateral patches. Since such quads need not be planar, in the bottom figure we have triangulated the quadrilaterals. A surface made using our method on 100 trajectories is shown in Figure 4. Compare with the surface shown in Figure 12 which was obtained by taking isochronous cross-sections. The distortion due to the points on different trajectories traveling at different speeds that was apparent in Figure 12 is not present in the surfaces shown in this figure, obtained by our method of planar cross-sections. One can still see the distortion that comes from the trajectories winding around the tube. Our method lays the groundwork to address this. Since we compute spline interpolations of the cross-section curves, as shown in Figures 2,3,5 these can be resampled from the highest point, to obtain patches whose edges along the tube do not follow the twisting of trajectories. We have not yet implemented this resampling. (The errors in hidden line and hidden surface removal in these surfaces near the self-intersections is a limitation of the rendering program used).

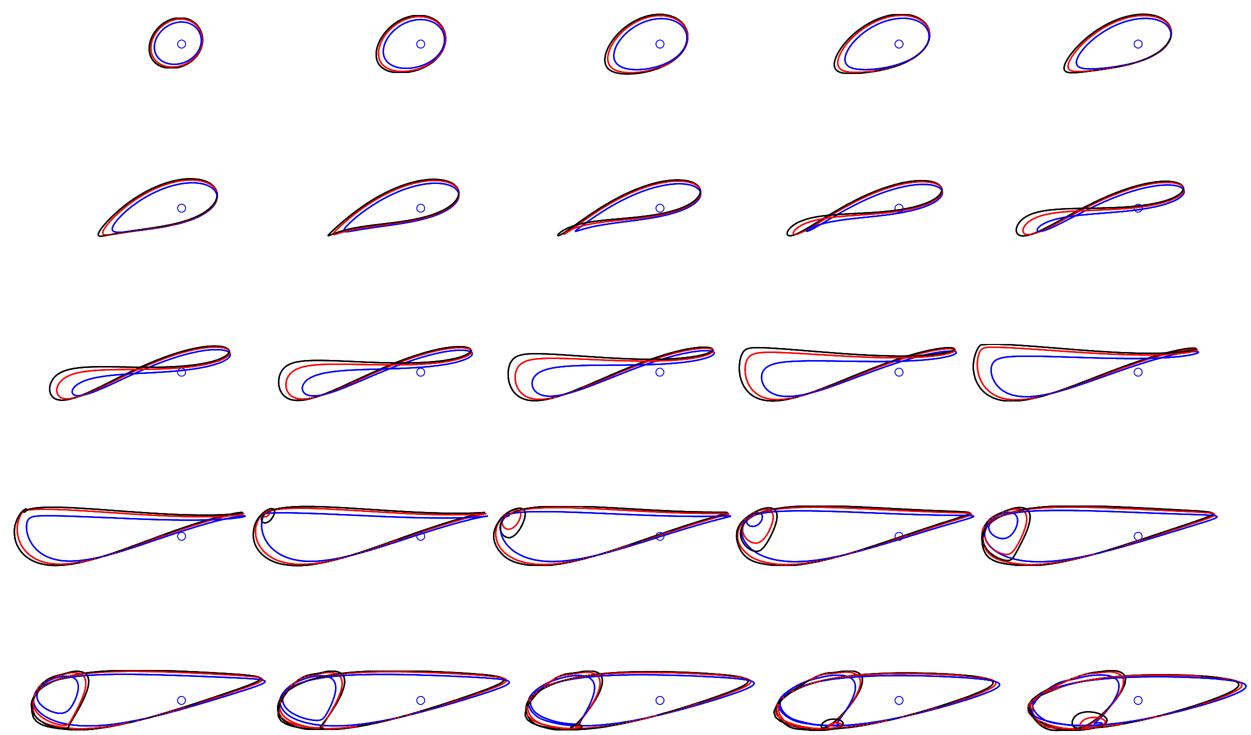

Figure 14: Cross sections of tubes of 3 energies showing that ordering of tubes is not maintained. The 3 colors correspond to tubes of the 3 different values of Jacobi constant. The curves are arranged left to right, top to bottom, as one moves downstream along the tube from the halo orbit that generates this tube. At first (top left) the blue tube is inside the red which is inside the black (this is not clear at this size, but see Figure 15 to verify this). The small blue circle is the location of the invariant manifold of the equilibrium point L_2 which is also the origin of the plane in which the cross section curves belong.

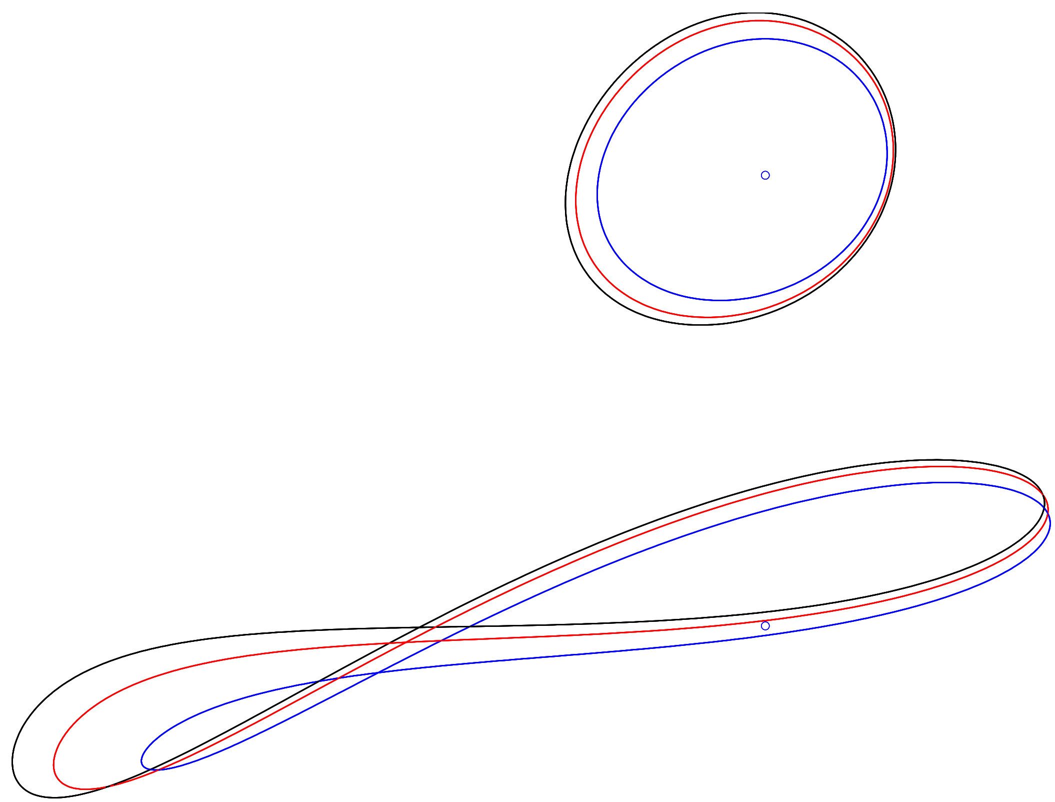

Figure 15: Close-up of two of the cross-sections of Figure 14. This shows more clearly that ordering of tubes is not maintained. The 3 colors correspond to tubes of the 3 different values of Jacobi constant. At first (top figure) the blue tube is inside the red which is inside the black. At some distance down the tube (bottom figure) parts of the cross-section curves have a different ordering. The small blue circle is the location of the invariant manifold of the equilibrium point L_2 which is also the origin of the plane in which the cross section curves belong.